

EIGENVALUES OF TOEPLITZ OPERATORS ON THE ANNULUS AND NEIL ALGEBRA

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ABSTRACT. By working with all collection of all the Sarason Hilbert Hardy spaces for the annulus algebra an improvement to the results of Aryana and Clancey on eigenvalues of self adjoint Toeplitz operators on an annulus is obtained. The ideas are applied to Toeplitz operators on the Neil algebra. These examples may provide a template for a general theory of Toeplitz operators with respect to an algebra.

1. INTRODUCTION

In this article, eigenvalues for Toeplitz operators with real symbols associated to the Neil algebra and the algebra of bounded analytic functions on an annulus are investigated.

The Neil algebra \mathcal{A} is the subalgebra of H^∞ consisting of those f whose derivative at 0 is 0. Pick interpolation in this, and other related more elaborate subalgebras of H^∞ , is a current active area of research with [DPRS] [BBtH] [JKM] [K] among the references.

For the algebra $A(\mathbb{A})$ of functions analytic on the annulus \mathbb{A} and continuous on the closure of \mathbb{A} the results obtained here give finer detail than those of Aryana and Clancey [AC] in their generalization of a result of Abrahamse [A1]. The proofs are accessible to readers familiar with basic functional analysis and function theory on the annulus as found in either [F] or [S]. The proofs don't involve theta functions.

The approach used and structure exposed here applies to many other algebras, including $H^\infty(R)$ for a (nice) multiply connected domain in \mathbb{C} and finite codimension subalgebras of H^∞ , though the details would necessarily be more complicated and less concrete than for the two algebras mentioned above.

The article proper is organized as follows. The algebras $A(\mathbb{A})$ and \mathcal{A} are treated in Sections 2 and 3 respectively. These sections can be read independently. Only the standard theory of H^2 is needed for Section 3. The article concludes with Section 4; it provides an additional rationale for

Date: March 15, 2013.

2010 Mathematics Subject Classification. 47Axx (Primary). 47B35 (Secondary).

Key words and phrases. Toeplitz operator, annulus, Neil parabola, bundle shift.

* I'd like to thank my advisor, Scott McCullough, for his advice and patience.

considering families of representations when studying Toeplitz operators associated to the algebras \mathcal{A} and $A(\mathbb{A})$.

2. TOEPLITZ OPERATORS ON THE ANNULUS

Fix $0 < q < 1$ and let \mathbb{A} denote the annulus,

$$\mathbb{A} = \mathbb{A}_q = \{z \in \mathbb{C} : q < |z| < 1\}.$$

The boundary B of \mathbb{A} has two components

$$B_q = \{z \in \mathbb{C} : |z| = q\}$$

and

$$B_1 = \{z \in \mathbb{C} : |z| = 1\}.$$

It is well known that for \mathbb{A} the analog of the classical Hilbert Hardy space H^2 on the disc is a one parameter family of Hilbert spaces that can be described in several different ways [AD],[S], or [A1]. For our purposes the following is convenient. Following [S] we will use the universal covering space of the annulus, $\widehat{\mathbb{A}} = \{(r, t) \in \mathbb{R}^2 : q < r < 1 \text{ and } -\infty < t < \infty\}$ and the map $\phi(r, t) \mapsto re^{it}$, to define modulus automorphic functions. A **Modulus Automorphic** function, F , on $\widehat{\mathbb{A}}$ is a meromorphic function on $\widehat{\mathbb{A}}$ that satisfies

$$|F(r, t)| = |F(r, t + 2n\pi)| \text{ for all } q < r < 1, 0 \leq t < 2\pi \text{ and } n \in \mathbb{Z}.$$

This means that although function $f := F \circ \phi^{-1}$ may be multivalued, $|f|$ is single valued. Because an analytic function is determined, up to a unimodular constant, by its modulus, if F is modulus automorphic, then there exists a constant of modulus one, λ_F , such that $F(r, t + 2\pi) \equiv \lambda_F F(r, t)$. The **index** of F , denoted by $\text{index}(F)$, is the unique $\alpha \in [0, 1)$ such that $\alpha = (2\pi i)^{-1} \log \lambda_F$. Let μ_j denote the multiple of arclength measure on B_j weighted so that $\mu_j(B_j) = 2\pi$ and let $\mu = \mu_1 + \mu_2$. Given $\alpha \in [0, 1)$, define an analog of $H^2(\mathbb{D})$ in the following way

$$H_\alpha^2(\mathbb{A}) := \{F \circ \phi^{-1} : \text{index}(F) = \alpha \text{ and } \int_B |F \circ \phi^{-1}|^2 d\mu < \infty\}.$$

In [S, Section 7] Sarason established the following important properties of $H_\alpha^2(\mathbb{A})$:

- (1) $H_\alpha^2(\mathbb{A}) \subseteq L^2(\mathbb{A})$ for all $\alpha \in [0, 1)$ and,
- (2) $H_\alpha^2(\mathbb{A}) = \{z^\alpha f : f \in H_0^2(\mathbb{A})\}.$

Moreover Sarason showed that the Laurent polynomials are dense in $H_0^2(\mathbb{A})$ and thus $H_0^2(\mathbb{A})$ admits an analog to Fourier Analysis on the disk.

Turning to multiplication and Toeplitz operators on the H_α^2 spaces, let $C(\overline{\mathbb{A}})$ denote the Banach algebra (in the uniform norm) of continuous functions on the closure of \mathbb{A} . The annulus algebra, $A(\mathbb{A})$, is the (Banach) subalgebra of $C(\overline{\mathbb{A}})$ consisting of those f which are analytic in \mathbb{A} . It's easy to

see that each H_α^2 space is invariant for $A(\mathbb{A})$ in the following sense that each $a \in A(\mathbb{A})$ determines a bounded linear operator M_a^α on H_α^2 defined by

$$M_a^\alpha f = af.$$

Moreover the mapping $\pi_\alpha : A(\mathbb{A}) \rightarrow B(H_\alpha^2)$ defined by $\pi_\alpha(a) = M_a^\alpha$ is a unital representation of the algebra $A(\mathbb{A})$ into the space $B(H_\alpha^2)$ of bounded linear operators on the Hilbert space H_α^2 .

Next let $\phi \in L^\infty$ denote a real valued function on B . The symbol ϕ determines a family, one for each α , of Toeplitz operators. Specifically, let T_ϕ^α denote the **Toeplitz** operator on H_α^2 defined by

$$H_\alpha^2 \ni f \mapsto P_\alpha \phi f,$$

where P_α is the projection of L^2 onto H_α^2 . A function $g \in H_\alpha^2$ is **outer** if $\{ag \in H_\alpha^2 : a \in A(\mathbb{A})\}$ is dense in the Hilbert space H_α^2 (see [S, Theorem 14, p. 62]).

The following is the main result on the existence of eigenvalues for Toeplitz operators on \mathbb{A} .

Theorem 2.1. *Fix a real valued $\phi \in L^\infty$. Let $\alpha \in [0, 1)$ and a nonzero $g \in H_\alpha^2$ be given. If $T_\phi^\alpha g = 0$, then g is outer and moreover there exists a nonzero $c \in \mathbb{R}$ such that*

$$(3) \quad \phi |g|^2 = c \log |zq^{-1/2}|.$$

If there is an α and an outer function $g \in H_\alpha^2$ such that Equation (3) holds, then $T_\phi^\alpha g = 0$, where α is necessarily the index of g . Thus α is congruent modulo 1 to,

$$(4) \quad \frac{1}{4\pi \log q} \left(\int_{B_1} \log |\phi| d\mu_1 - \int_{B_q} \log |\phi| d\mu_q \right)$$

In particular, there exists at most one α such that T_ϕ^α has eigenvalue 0 and the dimension of this eigenspace is at most one.

Before we prove Theorem 2.1, we pause to collect two corollaries. We say that λ is an **eigenvalue of ϕ relative to $A(\mathbb{A})$** if there exists an $\alpha \in [0, 1)$ and nontrivial solution g to $T_\phi^\alpha g = \lambda g$.

Corollary 2.2. *If*

$$\operatorname{essinf}\{\phi(z) : B_1\} = M > 0 > m = \operatorname{esssup}\{\phi(z) : B_q\}$$

or

$$\operatorname{essinf}\{\phi(z) : B_q\} = M > 0 > m = \operatorname{esssup}\{\phi(z) : B_1\},$$

then each $\lambda \in (m, M)$ is an eigenvalue of ϕ relative to $A(\mathbb{A})$, the latter case only happening when the c from Theorem 2.1 is negative. Further, M (resp. m) is an eigenvalue if and only if $\frac{\log |zq^{-1/2}|}{\phi - M} \in L^1$ (resp. $\frac{\log |zq^{-1/2}|}{\phi - m} \in L^1$).

Corollary 2.3. *The set of eigenvalues of ϕ relative to $A(\mathbb{A})$ is either empty, a point, or an interval.*

The following Corollary of Theorem 2.1 and Corollary 2.2 generalizes the main result of [AC] for the annulus.

Corollary 2.4. *With the hypotheses of Corollary 2.2 if either*

$$\int_B \log |\phi - M| = -\infty$$

or

$$\int_B \log |\phi - m| = -\infty,$$

then for each α the Toeplitz operator T_ϕ^α has infinitely many eigenvalues in the interval $(-m, M)$.

The remainder of this section is organized as follows: Subsection 2.1 contains the proof of Theorem 2.1, the corollaries are proved in Subsection 2.2.

2.1. Proof of Theorem 2.1. A function θ in H_β^2 is **inner** if $|\theta| = 1$ on B . A version of inner-outer factorization for the annulus is the following; given $f \in H_\alpha^2$, there is a β and an inner function $\psi \in H_\beta^2$ and outer function $F \in H_{\alpha-\beta}^2$ such that

$$(5) \quad f = \psi F.$$

Let

$$A(\mathbb{A})^* := \{f^* : f \in A(\mathbb{A})\}$$

and let $L^2 := L^2(B)$. Observe that the orthogonal complement in L^2 of $A(\mathbb{A}) \oplus A(\mathbb{A})^*$ is spanned by $\log |zq^{-1/2}|$.

In the context of Theorem 2.1, suppose $T_\phi^\alpha g = 0$. Using the fact that, if $a \in A(\mathbb{A})$, then $ag \in H_\alpha^2$ it follows that

$$\begin{aligned} 0 &= \langle T_\phi^\alpha g, ag \rangle \\ &= \int_B \phi |g|^2 a^* d\mu. \end{aligned}$$

Since $\phi |g|^2$ is real valued

$$\int_B \phi |g|^2 a d\mu = 0$$

too.

Lemma 2.5. *If $f \in L^1(B)$ is real-valued and*

$$\int_B f z^n = 0$$

for all $n \in \mathbb{Z}$, then there is a constant c so that $f = c \log |zq^{-\frac{1}{2}}|$.

Proof. For each $n \in \mathbb{Z}$

$$\begin{aligned}\int_B f z^n &= \sum_{j=0}^1 \int_0^{2\pi} f(q^j e^{it}) q^j e^{int} dt \text{ and} \\ \int_B f \bar{z}^n &= \sum_{j=0}^1 \int_0^{2\pi} f(q^j e^{it}) q^j e^{-int} dt.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{2\pi} f(e^{it}) e^{int} dt &= - \int_0^{2\pi} f(qe^{it}) q^n e^{int} dt \text{ and} \\ \int_0^{2\pi} f(e^{it}) e^{-int} dt &= - \int_0^{2\pi} f(qe^{it}) q^n e^{-int} dt.\end{aligned}$$

By replacing n with $-n$ in the last equation we can see that

$$q^n \int_0^{2\pi} f(qe^{it}) e^{int} dt = q^{-n} \int_0^{2\pi} f(qe^{it}) e^{int} dt.$$

Hence for all $n \in \mathbb{Z}$

$$(q^n - q^{-n}) \int f(q^j e^{it}) e^{int} dt = 0.$$

Which means that for all $0 \neq m \in \mathbb{Z}$ and $j = 0, 1$ we have that

$$\int_0^{2\pi} f(q^j e^{it}) e^{imt} dt = 0.$$

So f must be constant on each boundary with $a := f|_{B_1} = -f|_{B_q}$. If we choose $c \in \mathbb{R}$ to be $\frac{a}{\log q^{-\frac{1}{2}}}$ then $c \log |zq^{-\frac{1}{2}}| = f$ on B . ■

The next objective is to show that g is outer. To this end let $g = \psi F$ denote the inner-outer factorization of g as an H_α^2 function as in Equation (5) and let $\beta \in [0, 1)$ be the index of ψ . Since $\text{index}(\psi) + \text{index}(F) = \text{index}(g)$ we have that $z^\beta F \in H_\alpha^2$. Similarly we have that $C := z^{-\beta} \psi \in H^2$ which means that it has Fourier series for both the inner and outer boundaries, which we will denote by $\widehat{C}_q(n)$ and $\widehat{C}_1(n)$, moreover by [S, Lemma 1.1] we know that $\widehat{C}_1(n) = q^{-n} \widehat{C}_q(n)$. Thus for any $n \in \mathbb{Z}$

$$\begin{aligned}(6) \quad 0 &= \langle T_\phi^\alpha g, z^n z^\beta F \rangle \\ &= \int_B \phi |g|^2 |z|^{2\beta} z^{-\beta} \psi \bar{z}^n \\ &= \int_B \log(|zq^{-1/2}|) |z|^{2\beta} C \bar{z}^n \\ &= \log(q^{1/2}) (q^{n+2\beta} \widehat{C}_q(n) - \widehat{C}_1(n)) \\ &= \log(q) \widehat{C}_q(n) (q^{n+2\beta} - q^{-n}).\end{aligned}$$

From Equation (6) it follows that, for each n , either $\widehat{C}_q(n) = 0$ or $n + \beta = 0$. Since $\beta \in [0, 1)$ and $\widehat{C}_q(m) \neq 0$ for some m , it follows that $\widehat{C}_q(n) = 0$ for $n \neq 0$ and $\beta = 0$. Thus ψ is a unitary constant and g is outer.

To prove the second part of the Theorem, simply choose α to be the index of g . From [S, Theorem 6.6] we know that modulo 1 the index of g is

$$\frac{-1}{2\pi \log q} \left(\int_{B_1} \log |g| d\mu_1 - \int_{B_q} \log |g| d\mu_q \right).$$

Applying the fact that $|g| = \left(\frac{c \log |zq^{-1/2}|}{\phi} \right)^{1/2}$ the equation above simplifies to

$$\frac{1}{4\pi \log q} \left(\int_{B_1} \log |\phi| d\mu_1 - \int_{B_q} \log |\phi| d\mu_q \right).$$

Finally, suppose that $T_\phi^\alpha g = 0$ and also $T_\phi^\beta h = 0$. From what has already been proved g and h are outer and there exists nonzero $c, d \in \mathbb{R}$ such that

$$\phi |g|^2 = c \log |zq^{-1/2}|, \quad \phi |h|^2 = d \log |zq^{-1/2}|$$

on B . Since ϕ is almost everywhere nonzero we have that $|g|^2 = \frac{c}{d} |h|^2$ and because g and h are outer, they are equal up to a complex scalar multiple, see [S, Theorem 7.9].

2.2. Proofs of the corollaries. To prove Corollary 2.2, observe that the hypothesis imply, for $m < \lambda < M$ that

$$\psi = \frac{c \log |zq^{-1/2}|}{\phi - \lambda}$$

takes nonnegative values and is essentially bounded above and below away from zero. Hence by [S, Theorem 9] there exists an outer function g such that $|g|^2 = \psi$. From Theorem 2.1 there is a α such that $T_\phi^\alpha g = \lambda g$.

The case $\lambda = M$ (resp. $\lambda = m$) is similar, but now, while ψ will be essentially bounded below away from zero, it need not be integrable. If it is integrable, then the argument above shows ψ is an eigenvector with eigenvalue M . On the other hand, if M is an eigenvalue, then there is an outer function g so that $\psi = |g|^2$ and hence ψ is integrable.

From Corollary 2.2, the set of eigenvalues of ϕ relative to $A(\mathbb{A})$ contains the interval (m, M) . Thus to prove Corollary 2.3 it suffices to show if $M = \text{essinf}\{\phi(z) : z \in B_1\}$ and $\lambda > M$ (resp. $m = \text{esssup}\{\phi(z) : z \in B_q\}$ and $\lambda < m$), then λ is not an eigenvalue. To this end, suppose $\lambda > M$. Since $\lambda > M = \text{essinf}\{\phi(z) : z \in B_1\}$ we have that $\mu(\{\phi(z) - \lambda < 0 : z \in B_1\}) > 0$. On the other hand, if λ is an eigenvalue, then there is a nonzero $c \in \mathbb{R}$ and multivalued outer function $h \in L^2$ such that $(\phi - \lambda) |h|^2 = c \log |zq^{-1/2}|$ which implies that either $(\phi - \lambda)|_{B_q}$ is positive almost everywhere or $(\phi - \lambda)|_{B_1}$ is positive almost everywhere. This is a contradiction since we know that $(\phi - \lambda)$

is negative on the inner boundary of the annulus and not positive almost everywhere on the outer boundary.

It suffices to prove Corollary 2.4 when $M = \text{essinf}\{\phi(z) : z \in B_1\}$ and $\int_B \log |\phi(z) - M| d\mu = -\infty$, so rewrite equation (4) into the context of Corollary 2.2 to see that the index of g is congruent modulo one to

$$\beta_\lambda := \frac{1}{4\pi \log q} \left(\int_{B_1} \log |\phi(z) - \lambda| d\mu_1 - \int_{B_q} \log |\phi(z) - \lambda| d\mu_q \right).$$

Notice that as λ approaches M on B_q we have that $\int_{B_q} \log |\phi - \lambda| < \infty$ since $\phi \in L^\infty$ and $\text{esssup}\{\phi(z) : z \in B_q\} < M$. It follows from the monotone convergence theorem that β_λ approaches $-\infty$ as λ approaches M . Thus since the index, α , of the solution, g , to $T_\phi^\alpha = \lambda g$ is congruent modulo 1 to β_λ we see that each T_ϕ^α must have infinitely many eigenvalues.

3. TOEPLITZ OPERATORS ON THE NEIL PARABOLA

Let \mathcal{A} denote the Neil Algebra; i.e., \mathcal{A} is the unital subalgebra of the disc algebra $\mathbb{A}(\mathbb{D})$ consisting of those f with $f'(0) = 0$. Each subspace $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ determines a subspace

$$H_{\mathcal{V}}^2 = H^2 \ominus \mathcal{V}$$

of the classical Hardy space H^2 which is invariant for \mathcal{A} in the following sense. Each $a \in \mathcal{A}$ determines a bounded linear operator $M_a^\mathcal{V}$ on $H_{\mathcal{V}}^2$ defined by

$$M_a^\mathcal{V} f = af.$$

Moreover, the mapping $\pi_\mathcal{V} : \mathcal{A} \rightarrow B(H_{\mathcal{V}}^2)$ defined by $\pi_\mathcal{V}(a) = M_a^\mathcal{V}$ is a unital representation. Here $B(H_{\mathcal{V}}^2)$ is the algebra of bounded operators on $H_{\mathcal{V}}^2$. A further discussion of the collection representations $\pi_\mathcal{V}$ can be found in Section 4.

Let ϕ denote a real valued function on the unit circle \mathbb{T} . The symbol ϕ determines a family, one for each \mathcal{V} , of Toeplitz operators. Specifically, let $T_\phi^\mathcal{V}$ denote the **Toeplitz** operator on $H_{\mathcal{V}}^2$ defined by

$$H_{\mathcal{V}}^2 \ni f \mapsto P_\mathcal{V} \phi f,$$

where $P_\mathcal{V}$ is the projection of L^2 onto $H_{\mathcal{V}}^2$.

The following is the main result on the existence of eigenvalues for Toeplitz operators on $H_{\mathcal{V}}^2$.

Theorem 3.1. *Fix a real valued $\phi \in L^\infty$ and let $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ and nonzero $g \in H_{\mathcal{V}}^2$ be given. If $T_\phi^\mathcal{V} g = 0$, then g is outer and moreover there is a $c \in \mathbb{C}$ such that, on \mathbb{T} ,*

$$(7) \quad \phi |g|^2 = cz + (cz)^*.$$

Conversely, if there is a $c \in \mathbb{C}$ and outer function $g \in H^2$ such that Equation (7) holds, then $T_\phi^\mathcal{V} g = 0$, where \mathcal{V} is uniquely determined by the values $g(0)$ and $g'(0)$.

In particular, there exists at most one \mathcal{V} such that $T_\phi^\mathcal{V}$ has eigenvalue 0 and the dimension of this eigenspace is at most one.

Before turning to the proof of Theorem 3.1, we pause to state the analogs of corollaries 2.2, and 2.3. By analogy with the case of the annulus, we say that λ is an **eigenvalue of ϕ relative to \mathcal{A}** if there exists a $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ and nontrivial solution to $T_\phi^\mathcal{V} g = \lambda g$.

Corollary 3.2. Suppose ϕ , g , and c satisfy the conditions of Theorem 3.1 so that, in particular, Equation (7) holds. If

$$\operatorname{essinf}\{\phi(z) : cz + (cz)^* > 0\} = M > 0 > m = \operatorname{esssup}\{\phi(z) : cz + (cz)^* < 0\},$$

then each $\lambda \in (m, M)$ is an eigenvalue of ϕ relative to \mathcal{A} . Further, for each such λ there is an essentially unique outer function f_λ such that

$$(\phi - \lambda)|f_\lambda|^2 = cz + (cz)^*.$$

(Here c is independent of λ .)

Moreover, M (resp. m) is an eigenvalue if and only if $\frac{cz+(cz)^*}{\phi-M}$ (resp. $\frac{cz+(cz)^*}{\phi-m}$) is in L^1 .

Corollary 3.3. The set of eigenvalues of ϕ relative to \mathcal{A} is either empty, a point, or an interval.

If we are given a ϕ that satisfies Corollary 3.2 and a $\lambda \in (m, M)$ the following corollary allows us to determine exactly what $H_\mathcal{V}^2$ space the resulting outer function is in. But first for $z \in \mathbb{D}$ and t real, let

$$H(z, t) = \frac{e^{it} + z}{e^{it} - z}.$$

Corollary 3.4. Under the hypotheses of corollary 3.2 let

$$h_c := \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} H(z, t) \log |ce^{it} + c^* e^{-it}|^{1/2} dt\right),$$

and

$$g_\lambda := \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} H(z, t) \log |\phi(t) - \lambda|^{-1/2} dt\right).$$

The eigenvector f_λ of $T_\phi^\mathcal{V}$ associated with the eigenvalue λ is in the $H_\mathcal{V}^2$ space where (nontrivial) $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ is orthogonal to

$$h_c(0)g_\lambda(0) + (h_c(0)g'_\lambda(0) + h'_c(0)g_\lambda(0))z$$

Moreover since h_c and g_λ are outer functions neither $h_c(0)$ or $g_\lambda(0)$ are zero.

There is no analog of Corollary 2.4 or of the main result of [AC] for the Neil parabola. In fact Corollary 3.4 implies that if $e \subseteq \mathbb{C} \oplus \mathbb{C}z$ is spanned by 1 then no Toeplitz operator on H_e^2 has eigenvalues. Although an easier way to see that no Toeplitz operator on H_e^2 has eigenvalues is to

note that $H_e^2 = zH^2$. Moreover for similar reasons no Toeplitz operator on H_V^2 has eigenvalues if $V = \{0\}$ or $V = \mathbb{C} \oplus \mathbb{C}z$. In fact the following corollary says that there are many nonzero proper V , not just e , such that T_ϕ^V has no eigenvalues. To prove this we identify each nonzero proper V with an element of the complex projective line, $\mathbb{P}^1(\mathbb{C})$, which is $X = \mathbb{C}^2 \setminus \{0\}$ modulo the equivalence relation $v \sim w$ if and only if there is a complex number λ such that $v = \lambda w$. Let $\pi : X \rightarrow \mathbb{P}^1(\mathbb{C})$ denote the quotient mapping of $v \in \mathbb{C}^2 \setminus \{0\}$. The space $\mathbb{P}^1(\mathbb{C})$ can be realized as a Riemann surface by the charts $\Psi_j : \mathbb{C} \rightarrow \mathbb{P}^1(\mathbb{C})$ defined by $\Psi_0(\zeta) = (\zeta \ 1)^T$ and $\Psi_1(\xi) = (1 \ \xi)^T$. Indeed, the transition mappings between these charts are $\zeta = \frac{1}{\xi}$ and $\xi = \frac{1}{\zeta}$. A map $F : \mathbb{R} \rightarrow \mathbb{P}^1(\mathbb{C})$ is differentiable if the maps $\Psi_0^{-1} \circ F$ and $\Psi_1^{-1} \circ F$ are differentiable where defined. Finally, if I is an interval in \mathbb{R} and $g : I \rightarrow X$ is differentiable, then so is $\pi \circ g$ and in this case the Hausdorff dimension of the range of $\pi \circ g$ is one and in this sense the range is a relatively small subset of $\mathbb{P}^1(\mathbb{C})$.

Corollary 3.5. *The function, Λ , from (m, M) to $\mathbb{P}^1(\mathbb{C})$ defined by*

$$\lambda \mapsto \pi \left(\begin{pmatrix} h_c(0)g_\lambda(0) \\ h_c(0)g_\lambda(1) + h'_c(0)g'_\lambda(0) \end{pmatrix} \right)$$

is twice differentiable with respect to λ on (m, M) . Thus in addition to $V = \text{span}\{1\}$ there exist nonzero proper $V \subseteq \mathbb{C} \oplus \mathbb{C}z$ such that T_ϕ^V has no eigenvalues.

The remainder of this section is organized as follows. Subsection 3.1 contains the proof of Theorem 3.1. The corollaries are proved in Subsection 3.2.

3.1. Proof of Theorem 3.1. Let

$$\mathcal{A}^* := \{f^* : f \in \mathcal{A}\}$$

and let $L^2 := L^2(\mathbb{T})$ where \mathbb{T} is the unit circle. Observe that orthogonal complement in L^2 of the span of $\mathcal{A} \oplus \mathcal{A}^*$ is spanned by z and z^* .

In the context of Theorem 3.1, suppose $T_\phi^V g = 0$. Using the fact that, if $a \in \mathcal{A}$, then $ag \in H_V^2$ it follows that

$$\begin{aligned} 0 &= \langle T_\phi^V g, ag \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} \phi |g|^2 a^* dt. \end{aligned}$$

Since $\phi |g|^2$ is real valued it is also the case that

$$\frac{1}{2\pi} \int_0^{2\pi} \phi |g|^2 a dt = 0.$$

Hence for any integer n different from ± 1 the corresponding Fourier coefficient of $\phi |g|^2$ is zero. Thus there exists $c \in \mathbb{C}$ such that all the Fourier coefficients of $\phi |g|^2 - cz + (cz)^*$ are 0, which implies that $\phi |g|^2 = cz + (cz)^*$ almost everywhere on \mathbb{T} .

The next objective is to show g is outer. To this end, let $g = \Psi F$ denote the inner-outer factorization of g as an H^2 function. Observe that, $z^n F \in H_{\mathcal{V}}^2$ for integers $n \geq 2$. Thus, for such n ,

$$\begin{aligned} 0 &= \langle T_{\phi}^n g, F z^n \rangle \\ &= \frac{1}{2\pi} \int \phi |g|^2 \Psi \bar{z}^n \, dt \\ &= \langle (cz + (cz)^*) \Psi, z^n \rangle_{L^2}. \end{aligned}$$

It follows, writing $\Psi = \sum_{k=0}^{\infty} \Psi_k z^k$, that

$$c\Psi_{n-1} + c^* \Psi_{n+1} = 0$$

for $n \geq 2$. In particular,

$$\Psi_{2k+1} = \left(-\frac{c}{c^*}\right)^k \Psi_1$$

for $k \geq 1$ and likewise,

$$\Psi_{2k+2} = \left(-\frac{c}{c^*}\right)^k \Psi_2.$$

Because $\Psi \in H^2$ these last two equations imply that $\Psi_k = 0$ for $k \geq 1$; i.e., Ψ is a unimodular constant and thus g is outer, and the first part of the Theorem is established.

The proof of the converse uses the following lemma.

Lemma 3.6. *Given a nonzero $\mathcal{V} \subsetneq \mathbb{C} \oplus \mathbb{C}z$, if g is outer and in $H_{\mathcal{V}}^2$, then the set*

$$\{ag : a \in \mathcal{A}\}$$

is dense in $H_{\mathcal{V}}^2$.

Proof. Let $f \in H_{\mathcal{V}}^2$ be given. Since g is outer there exists a sequence of functions $\{a_n\} \subseteq H^\infty(\mathbb{D})$ such that $a_n g$ converges to f in $H^2(\mathbb{D})$. Let $b_n := a_n - a'_n(0)$, so each $b_n \in \mathcal{A}$. To show that $b_n g$ also converges to f we will first show that $a'_n(0)$ converges to zero. To do this note that for each nonzero proper \mathcal{V} there exists a pair $(\alpha, \beta) \in \mathbb{C}^2$ not both zero such that if $h \in H_{\mathcal{V}}^2$, then there exists a $\zeta \in \mathbb{C}$ and $q \in H^2(\mathbb{D})$ such that $h = \zeta\alpha + \zeta\beta z + z^2 q$. Since $g(0) \neq 0$ this means that

$$\frac{f(0)}{g(0)} \cdot g'(0) = f'(0).$$

Because $a_n(0)g(0)$ converges to $f(0)$ and $(a_n(0)g(0))'$ converges to $f'(0)$ we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n(0) &= \frac{f(0)}{g(0)} \\ \lim_{n \rightarrow \infty} a'_n(0) &= \frac{f'(0) - \frac{f(0)}{g(0)} \cdot g'(0)}{g(0)} = 0 \end{aligned}$$

Thus $a'_n(0)$ converges to 0 so $b_n g$ also converges to f . ■

To prove the converse, suppose that g is outer and there is a $c \in \mathbb{C}$ such that Equation (7) holds. Let \mathcal{V} be the nonzero subspace of $\mathbb{C} \oplus \mathbb{C}z$ such that $g(0) + g'(0)z \in \mathcal{V}^\perp$, then $g \in H_{\mathcal{V}}^2$. It follows that, for any $a \in \mathcal{A}$,

$$\langle T_\phi^\mathcal{V} g, ag \rangle = \int \phi |g|^2 a^* = 0$$

and thus, in view of Lemma 3.6, $T_\phi^\mathcal{V} g = 0$.

Finally, suppose that $T_\phi^\mathcal{V} g = 0$ and $T_\phi^\mathcal{W} h = 0$. From what has already been proved g and h are outer and there exists $c, d \in \mathbb{C}$ such that

$$\phi |g|^2 = cz + (cz)^*, \quad \phi |h|^2 = dz + (dz)^*$$

on \mathbb{T} . It follows that ϕ is positive almost everywhere both where $cz + (cz)^*$ and $(dz) + (dz)^*$ are positive. Hence $c = td$ for some positive real number t . But then, $t|g| = |h|$ and because g and h are outer, they are equal up to a (complex) scalar multiple.

3.2. Proofs of the corollaries. To prove Corollary 3.2, observe that the hypotheses imply, for $m < \lambda < M$ that

$$\psi = \frac{cz + (cz)^*}{\phi - \lambda}$$

takes nonnegative values, is in L^1 and moreover

$$(8) \quad \int \log |\psi| > -\infty$$

because the same is true with ψ replaced by $cz + (cz)^*$ and ϕ is essentially bounded. Hence there is an outer function $g \in H^2$ such that

$$(\phi - \lambda) |g|^2 = cz + (cz)^*.$$

From Theorem 3.1 there is a nonzero proper \mathcal{V} such that

$$T_\phi^\mathcal{V} g = \lambda g.$$

The case $\lambda = M$ (resp. $\lambda = m$) are similar, with the only issue being that a hypothesis is needed to guarantee that ψ , as defined above, is integrable.

Turning to the proof of Corollary 3.3, because Corollary 3.2 implies the interval (m, M) is contained in the set of eigenvalues of ϕ with respect to \mathcal{A} , it suffices to show if $\lambda > M$ (resp. $\lambda < m$), then λ is not an eigenvalue. Accordingly suppose $\lambda > M$. In this case the measure of the set $S = \{z \in \mathbb{T} : \phi(z) > \lambda\} < \frac{\pi}{2}$. On the other hand, if λ is an eigenvalue, then there is a non-zero c and outer function $h \in H^2$ such that

$$(\phi - \lambda) |h|^2 = cz + (cz)^*$$

But then the measure of the set S is $\frac{\pi}{2}$, a contradiction. It now follows that set of eigenvalues contains (m, M) and is contained in $[m, M]$ and the proof of the corollary is complete.

To prove corollary 3.4 use (7) and the fact that f_λ is outer to see that

$$f_\lambda = \exp \left(\int_{\mathbb{T}} H \log \left(\frac{cz + (cz)^*}{\phi - \lambda} \right)^{1/2} \right) = \exp \left(\int_{\mathbb{T}} H \log |cz + (cz)^*|^{1/2} \right) \exp \left(\int_{\mathbb{T}} H \log |\phi - \lambda|^{-1/2} \right) = h_c g_\lambda.$$

Thus $f_\lambda(0) = h_c(0)g_\lambda(0)$ and $f'_\lambda(0) = h_c(0)g'_\lambda(0) + h'_c(0)g_\lambda(0)$ and the conclusion follows.

To prove 3.5 it will suffice to show that the maps

$$\lambda \mapsto g_\lambda(0) \text{ and}$$

$$\lambda \mapsto g'_\lambda(0)$$

are differentiable with respect to λ on (m, M) . Those questions boil down to checking if

$$\lambda \mapsto \int_0^{2\pi} H(0, e^{it}) \log |\phi(e^{it}) - \lambda| \, dt \text{ and}$$

$$\lambda \mapsto \int_0^{2\pi} H'(0, e^{it}) \log |\phi(e^{it}) - \lambda| \, dt$$

are differentiable with respect to λ on (m, M) . Since for each $\lambda \in (m, M)$ we know that $\phi - \lambda$ is essentially bounded away from zero given any $B(\lambda, \delta) \subseteq (m, M)$ we can find a $N_{\lambda, \delta}$ such that for any $\sigma \in B(\lambda, \delta)$ we have $N_{\lambda, \delta} < |\phi - \sigma|$. So

$$\int_0^{2\pi} H(0, e^{it}) \log |\phi(e^{it}) - \lambda| \, dt \leq N_{\lambda, \delta}^{-1} \text{ and}$$

$$\int_0^{2\pi} H'(0, e^{it}) \log |\phi(e^{it}) - \lambda| \, dt \leq N_{\lambda, \delta}^{-1}$$

thus by the dominated convergence theorem both maps are differentiable with respect to λ on (m, M) . In fact for the same reason both are infinitely differentiable with respect to λ on (m, M) . Since $\Psi_1 \circ \Lambda$ is twice differentiable on (m, M) the map preserves the Hausdorff measure of (m, M) thus the Hausdorff dimension of $\Psi_1 \circ \Lambda((m, M))$ is 1. So $\Lambda((m, M))$ cannot be all of $\mathbb{P}^1(\mathbb{C}) \setminus \{[0, 1]\}$ since Ψ_1 is injective on its range. Finally for each nonzero proper \mathcal{V} choose a $f \in H_{\mathcal{V}}^2$ with $f(0)$ and $f'(0)$ both not zero and identify \mathcal{V} with $[\frac{f'(0)}{f(0)}, 1] \in \mathbb{P}^1(\mathbb{C})$ or $[1, \frac{f'(0)}{f(0)}] \in \mathbb{P}^1(\mathbb{C})$. Thus we have shown that in addition to $\mathcal{V} = \text{span}\{1\}$ there exist nonzero proper $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ such that $T_\phi^\mathcal{V}$ has no eigenvalues.

4. BUNDLE SHIFTS

It is natural to ask what distinguishes the families of representations $\{\pi_\alpha : 0 \leq \alpha < 1\}$ and $\{\pi_\mathcal{V} : \mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z\}$ of the algebras $A(\mathbb{A})$ and \mathcal{A} as multiplication operators on the spaces $\{H_\alpha^2 : 0 \leq \alpha < 1\}$ and $\{H_\mathcal{V}^2 : \mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z\}$ respectively.

For the annulus, an answer is that Sarason recognized that the collection of representations (π_α, H_α^2) played the same role on the annulus as the single representation determined by the shift

operator S given by

$$(9) \quad \mathbb{A}(\mathbb{D}) \ni f \rightarrow f(S)$$

plays for the disc algebra $\mathbb{A}(\mathbb{D})$. They also correspond to the rank one bundle shifts of Abrahamse and Douglas [AD] and generate a minimal family of positivity conditions sufficient for Pick interpolation on the annulus [A2] [BC]. For \mathcal{A} the representations $\pi_{\mathcal{V}}$ generate a family of positivity conditions sufficient for Pick interpolation in \mathcal{A} [DPRS]. Likely it is a minimal set of conditions too. Corollary 4.2 below can be interpreted as saying that the representations $(\pi_{\mathcal{V}}, H_{\mathcal{V}}^2; \mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z)$ should play the role of the rank one bundle shifts for the Neil algebra \mathcal{A} .

For positive integers n , the algebra $M_n(B(H))$ of $n \times n$ matrices with entries from $B(H)$ is naturally identified with $B(\mathbb{C}^n \otimes H)$, the operators on the Hilbert space $\mathbb{C}^n \otimes H \equiv \oplus_1^n H$. In particular, it is then natural to give an element $X \in M_n(B(H))$ the norm $\|X\|_n$ it inherits as an operator on $\mathbb{C}^n \otimes H$. If A is a subalgebra of $B(H)$, then the norms $\|\cdot\|_n$ of course restrict to $M_n(A)$, the $n \times n$ matrices with entries from A , and A together with this sequence of norms is a concrete **operator algebra**.

Turning to the disc algebra, an element $F \in M_n(\mathbb{A}(\mathbb{D}))$ takes the form $F = (F_{j,k})_{j,k=1}^n$ for $F_{j,k} \in \mathbb{A}(\mathbb{D})$. In particular, $M_n(\mathbb{A}(\mathbb{D}))$ is itself an algebra and comes naturally equipped with the norm,

$$\|F\|_n = \sup\{\|F(z)\| : z \in \mathbb{D}\}.$$

where $\|F(z)\|$ is the usual **operator norm** of the $n \times n$ matrix $F(z)$. The representation $\pi : \mathbb{A}(\mathbb{D}) \mapsto B(H^2)$ given by $\pi(a) = M_a$ extends naturally to $M_n(\mathbb{A}(\mathbb{D}))$ as $1_n \otimes \pi : M_n(\mathbb{A}(\mathbb{D})) \rightarrow B(\oplus^n H^2)$ by

$$1_n \otimes \pi(F) = \left(\pi(F_{j,k}) \right)_{j,k=1}^n.$$

Moreover, the maps $1_n \otimes \pi$ are isometric. Thus the algebra $\mathbb{A}(\mathbb{D})$ can be viewed as an operator algebra by identifying $\mathbb{A}(\mathbb{D})$ together with the sequence of norms $(\|\cdot\|_n)$ with its image in $B(H^2)$ under the mappings $1_n \otimes \pi$. Of course, any subalgebra of $\mathbb{A}(\mathbb{D})$ can also then be viewed as an operator algebra by inclusion.

Given an operator algebra A , a representation $\rho : A \rightarrow B(H)$ is **completely contractive** if $\|1_n \otimes \rho(F)\|_n \leq \|F\|_n$ for each n and $F \in M_n(A)$. If A and B are unital, then ρ is a **unital representation** if $\rho(1) = 1$. The representation ρ on $B(H)$ is **pure** if

$$\bigcap_{a \in A} \rho(a)H = (0).$$

It is immediate that the representations of $\mathbb{A}(\mathbb{D})$ determined by S as well as the representations π_{α} of $A(\mathbb{A})$ and $\pi_{\mathcal{V}}$ of \mathcal{A} are unital, completely contractive, and pure.

Following Agler [Ag], a completely contractive (unital) representation $\pi : \mathcal{A} \rightarrow B(H)$ of \mathcal{A} on the Hilbert space H is **extremal** if $\rho : \mathcal{A} \rightarrow B(K)$ is a completely contractive representation on

the Hilbert space K and $V : H \rightarrow K$ is an isometry such that

$$\pi(a) = V^* \rho(a) V$$

implies in fact that

$$V\pi(a) = \rho(a)V.$$

Given a Hilbert space \mathcal{N} , let $H_{\mathcal{N}}^2$ denote the Hilbert Hardy space of \mathcal{N} -valued analytic functions on the disc with square integrable boundary values. Associated to \mathcal{N} is the representation $\rho : \mathbb{A}(\mathbb{D}) \rightarrow B(H_{\mathcal{N}}^2)$ defined by

$$\rho(\varphi)f = \varphi f.$$

Thus, $\rho(\varphi)$ is multiplication by the scalar-valued φ on the vector-valued H^2 space $H_{\mathcal{N}}^2$. Of course, $H_{\mathcal{N}}^2$ is naturally identified with $\mathcal{N} \otimes H^2$ and the representation ρ is then the identity on \mathcal{N} tensored with the representation of $\mathbb{A}(\mathbb{D})$ in Equation (9). If ρ is a completely contractive unital pure extremal representation of $\mathbb{A}(\mathbb{D})$, then there exists a Hilbert space \mathcal{N} so that, up to unitary equivalence, $\rho : \mathbb{A}(\mathbb{D}) \rightarrow B(H_{\mathcal{N}}^2)$ is given by $\rho(\varphi)f = \varphi f$.

For \mathcal{A} it turns out that the subspaces of $H_{\mathcal{N}}^2$ identified in [R] give rise to the extremal representations. Indeed, given a Hilbert space \mathcal{N} and a subspace \mathcal{V} of the subspace $\mathcal{N} \oplus z\mathcal{N}$ of $H_{\mathcal{N}}^2$, the mapping $\pi_{\mathcal{V}} : \mathcal{A} \rightarrow B(H_{\mathcal{N}}^2 \ominus \mathcal{V})$ defined by

$$\pi_{\mathcal{V}}(a)f = af,$$

is easily seen to be a unital pure completely contractive representation of \mathcal{A} .

Theorem 4.1. *The representations $\pi_{\mathcal{V}}$ are unital pure completely contractive extremal representations. Moreover, if ν is a unital pure extremal completely contractive representation of \mathcal{A} , then ν is unitarily equivalent to $\pi_{\mathcal{V}}$ for some Hilbert space \mathcal{N} and $\mathcal{V} \subseteq \mathcal{N} \oplus \mathcal{N}z$.*

Finally we will say a representation has rank one if there do not exist an orthogonal pair of invariant subspaces.

Corollary 4.2. *The representations $\pi_{\mathcal{V}}$ for $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ have rank one. Moreover if the representation π is a unital pure extremal completely contractive rank one representation of \mathcal{A} , then there is a $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ such that π is unitarily equivalent to $\pi_{\mathcal{V}}$.*

The remainder of the section is organized as follows. Subsection 4.1 proves that the representations $\pi_{\mathcal{V}}$ are extremal. Subsection 4.2 contains the proof of the remainder of Theorem 4.1. The corollary is proved in 4.3.

4.1. The Extremal Representations of \mathcal{A} . While it is easy to see that the representations $\pi_\gamma : \mathcal{A} \rightarrow B(H_\gamma^2)$ are unital, pure, and completely contractive showing that they are also extremal is a bit harder. To prove they are extremal we will first prove a proposition which gives us an easy to verify sufficient condition for a representation to be extremal.

The first lemma we need is a well known generalization of Sarason's Lemma [S2]. Given a representation $\rho : A \rightarrow B(K)$, a subspace \mathcal{M} of K is **invariant** for ρ if $\rho(a)\mathcal{M} \subseteq \mathcal{M}$ for all $a \in A$. A subspace H of K is **semi-invariant** for ρ if there exist invariant subspaces \mathcal{M} and \mathcal{N} such that $H = \mathcal{N} \oplus \mathcal{M}$. Note that, letting $V : H \rightarrow K$ denote an isometry, the mapping $A \ni a \mapsto V\rho(a)V^*$ is also a representation of A .

Lemma 4.3. *Let $\nu : A \rightarrow B(H)$ be a representation of A in $B(H)$ and $\rho : A \rightarrow B(K)$ be a representation of A in $B(K)$ and $V : H \rightarrow K$ an isometry. If $\nu(a) = V^*\rho(a)V$ for all $a \in A$, then VH is a semi-invariant for ρ .*

Proof. Let

$$\mathcal{N} := \bigvee_{a \in A} \rho(a)VH.$$

Notice that the elements of the form

$$\sum_{i=0}^N \rho(a_i)Vh_i \text{ where } \{h_i\} \subseteq H, \{a_i\} \subseteq A, \text{ and } N > 0$$

form a dense subset of \mathcal{N} . Since ρ is a representation, for any $a \in A$, $\{h_i\} \subseteq H$, $\{a_i\} \subseteq A$, and $N > 0$ we have that

$$\rho(a) \left(\sum_{i=0}^N \rho(a_i)Vh_i \right) = \sum_{i=0}^N \rho(a \cdot a_i)Vh_i \in \mathcal{N}$$

and thus \mathcal{N} is $\rho(a)$ invariant. To complete this direction we only need to show that $\mathcal{M} := \mathcal{N} \oplus VH$ is invariant for $\rho(a)$. Notice that $\pi(a) = V^*\rho(a)V$ implies

$$\begin{aligned} V^*\rho(a) \left(\sum_{i=0}^N \rho(a_i)Vh_i \right) &= \sum_{i=0}^N V^*\rho(a \cdot a_i)Vh_i \\ &= \sum_{i=0}^N \nu(a)\nu(a_i)h_i \\ &= \nu(a) \sum_{i=0}^N V^*\rho(a_i)Vh_i. \end{aligned}$$

Thus $V^*\rho(a)|_{\mathcal{N}} = \nu(a)V^*|_{\mathcal{N}}$. If $m \in \mathcal{M}$, then $m \in \mathcal{N}$ and by the Fredholm alternative $V^*m = 0$. Thus, if $a \in A$, then $V^*\rho(a)m = \nu(a)V^*m = 0$, which, again by the Fredholm alternative, implies $\rho(a)m \in \mathcal{M}$. ■

The second allows us to improve semi-invariance to invariant if $\nu(a)$ is an isometry and $\|\rho(a)\| = 1$.

Lemma 4.4. *If H is a semi-invariant subspace for a contraction T with \mathcal{N} and \mathcal{M} both T invariant, $\mathcal{N} = H \oplus \mathcal{M}$, and $S := P_H T|_H$ is an isometry, then H is an invariant subspace for T .*

Proof. Since H is semi-invariant for T and $S = P_H T|_H$ we know that there exists two T invariant spaces \mathcal{N} and \mathcal{M} such that $\mathcal{N} = \mathcal{N} \oplus H$ and

$$T = \begin{bmatrix} A & B & C \\ 0 & S & F \\ 0 & 0 & K \end{bmatrix}$$

Where $A : \mathcal{M} \rightarrow \mathcal{M}$, $B : \mathcal{M} \rightarrow H$, $C : \mathcal{M} \rightarrow \mathcal{N}^\perp$, $F : H \rightarrow \mathcal{N}^\perp$, and $K : \mathcal{N}^\perp \rightarrow \mathcal{N}^\perp$. Since T is a contraction we have $I - T^*T \geq 0$ thus for all $h \in H$, thus

$$0 \leq \left\langle (I - T^*T) \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -A^*Bh \\ (I - B^*B - S^*S)h \\ -(C^*B + F^*S)h \end{bmatrix}, \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix} \right\rangle = \|h\|^2 - \|Bh\|^2 - \|Sh\|^2 = -\|Bh\|^2$$

so $Bh = 0$ for all $h \in H$. Thus H is a T invariant subspace. ■

Combining Lemmas 4.3 and 4.4 yields the following proposition.

Proposition 4.5. *Let $\nu : A \rightarrow B(H)$ be a contractive representation of A in $B(H)$ and $\{a_i\}_{i \in J} \subseteq A$ be a set that generates a dense subalgebra of A with $\|a_i\| = 1$ for all $i \in J$. If $\nu(a_i)$ is an isometry for all $i \in J$, then ν is extremal.*

Proof. Let $\rho : A \rightarrow B(K)$ be a contractive representation of A in $B(K)$ and $V : H \rightarrow K$ be an isometry such that $\nu(a) = V^* \rho(a) V$ for all $a \in A$. By lemma 4.3 we have that VH is a semi-invariant subspace of K for $\rho(a)$ and any $a \in A$. Let $\{a_i\}_{i \in J} \subseteq A$ generate A , then by lemma 4.4 we have that VH is invariant for $\rho(a_i)$ for each $i \in J$. Because ρ is a representation we have that VH is invariant for $\rho(a)$ where

$$a \in A^\circ := \left\{ \sum_{i=0}^N c_i \prod_{j=0}^M a_{k_{i,j}} \mid c_i \in \mathbb{C}, N, M \in \mathbb{N}, \text{ and } k_{i,j} \in J \right\}.$$

Since A° set is dense in A and ρ is bounded VH must be invariant for all $a \in A$.

Now since VH is $\rho(a)$ invariant and $\nu(a) = V^* \rho(a) V$ for all $a \in A$ we have that

$$V \nu(a) = V V^* \rho(a) V = P_{VH} \rho(a) V = \rho(a) V \text{ for all } a \in A.$$

Thus ν is extremal. ■

Now it is easy to show that all of the π_ν 's are extremal representations of \mathcal{A} .

Corollary 4.6. *The representation $\pi_{\mathcal{V}}$ is an extremal representation of \mathcal{A} .*

Proof. Since $\pi_{\mathcal{V}}(1)$, $\pi_{\mathcal{V}}(z^2)$, and $\pi_{\mathcal{V}}(z^3)$ are isometries and 1 , z^2 , and z^3 generate \mathcal{A} , by proposition 4.5 we know that $\pi_{\mathcal{V}}$ is extremal. ■

4.2. Proof of Theorem 4.1. Let $\nu : \mathcal{A} \rightarrow B(H)$ be a pure extremal representation of \mathcal{A} on some separable Hilbert space H . By [P, Corollary 7.7], the representation ν has an $C(\mathbb{T})$ -dilation; i.e., there exists a completely contractive representation $\rho : L^\infty(\mathbb{D}) \rightarrow B(K)$ and an isometry $V : H \rightarrow K$ such that $\rho(a) = V^* \nu(a) V$ for all $a \in \mathcal{A}$. Moreover since ν is extremal $V \nu(a) = \rho(a) V$ for all $a \in \mathcal{A}$. Finally let

$$E = \bigvee_{i=0}^{\infty} \rho(z^i) V H \subseteq K.$$

Since $z^i \in \mathcal{A}$ for all $i \in \mathbb{N}$ and $i \neq 1$,

$$\bigvee_{\substack{i=0 \\ i \neq 1}}^{\infty} \rho(z^i) V H = \bigvee_{\substack{i=0 \\ i \neq 1}}^{\infty} V \nu(z^i) H = V H.$$

In particular, $E = \rho(z) V H \vee V H$.

First we will show that $S = \rho(z)|_E$ is a pure isometry on E ; if $f, g \in K$, then

$$\langle \rho(z)f, \rho(z)g \rangle = \langle \rho(z)^* \rho(z)f, g \rangle = \langle \rho(\bar{z}z)f, g \rangle = \langle f, g \rangle.$$

Since S is the restriction of an isometry to an invariant subspace S is an isometry. To show that S is pure note that

$$\rho(z^2)E = \rho(z^2)(\rho(z)VH \vee VH) = \rho(z^3)VH \vee \rho(z^2)VH \subseteq VH.$$

Since ν is pure we have

$$\begin{aligned} \bigcap_{b \in \mathbb{A}(\mathbb{D})} \rho(b)E &\subseteq \bigcap_{\substack{b=z^2a \\ a \in \mathcal{A}}} \rho(a)\rho(z^2)E \\ &\subseteq \bigcap_{a \in \mathcal{A}} \rho(a)VH \\ &= \bigcap_{a \in \mathcal{A}} V \nu(a)H = \{0\}. \end{aligned}$$

Thus S is a pure shift on E .

Since S is a pure shift there is a Hilbert space \mathcal{N} and a unitary map $W : E \rightarrow H_{\mathcal{N}}^2$ such that $WS = M_z W$. Since the subspace $S^2 E$ lies in VH and VH is a subspace of E , there exists a subspace \mathcal{V} of $\mathcal{N} \oplus z\mathcal{N}$ such that $WVH = H_{\mathcal{V}}^2 = H_{\mathcal{N}}^2 \ominus \mathcal{V}$. Let $U : H \rightarrow H_{\mathcal{V}}^2$ be defined by $U = WV$, this is a unitary map such that

$$U^* \pi_{\mathcal{V}}(a) U h = U^* M_a U h = \nu(a)h \text{ for all } h \in H,$$

i.e. $\pi_{\mathcal{V}}$ is unitarily equivalent to ν .

4.3. Proof of Corollary 4.2. Suppose $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ and M and N are orthogonal subspaces of $H_{\mathcal{V}}^2$ invariant for \mathcal{A} . Choosing non-zero φ and ψ from M and N respectively, it follows that $\langle z^m \varphi, z^n \psi \rangle = 0$ for natural numbers $m \neq 1 \neq n$. Hence,

$$0 = \int \varphi \bar{\psi} z^j dt$$

for all j and therefore $\varphi \bar{\psi} = 0$. Since both φ and ψ are in H^2 , they are non-zero almost everywhere. Thus at least one must be zero, which is a contradiction. So if $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ then $\pi_{\mathcal{V}}$ is rank one.

By theorem 4.1 it suffices to check the second part of the corollary for $\pi_{\mathcal{V}}$ where $\mathcal{V} \subseteq \mathcal{N} \oplus \mathcal{N}z$. If \mathcal{N} is one dimensional then \mathcal{N} is unitarily equivalent to \mathbb{C} and we are done. If \mathcal{N} is not one dimensional, then choose a pair of non-zero vectors e and f in \mathcal{N} such that $\langle e, f \rangle = 0$ and let $\mathcal{E} = z^2 H^2 e$ and $\mathcal{F} = z^2 H^2 f$. Both \mathcal{E} and \mathcal{F} are non-trivial subspaces of $H_{\mathcal{V}}^2$ for any \mathcal{V} and are \mathcal{A} invariant. They are also orthogonal by construction. Hence $\pi_{\mathcal{V}}$ is not rank one.

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